Math 324 Spring 2017
Homework 9
Due: April 26, 2017

---

1. Let K be an algebraic number field. Prove that $\mathcal{O}_K$ contains infinitely many prime ideals.

2. We can use the trace to show that a complex number is not in a particular number field. This problem will works you through the steps to show $\sqrt{3} \notin \mathbb{Q}(\sqrt[4]{2})$. We use the fact that as a vector space, $\{1, \sqrt[4]{2}, (\sqrt[4]{2})^2, (\sqrt[4]{2})^3\}$ is a basis of K over $\mathbb{Q}$.

   (a) Suppose, by way of contradiction, that $\sqrt{3} = a + b\sqrt[4]{2} + c\sqrt{2} + d\sqrt[4]{8}$ with a, b, c, $d \in \mathbb{Q}$. Compute the trace of both sides and conclude that $a = 0$.

   (b) This means $\sqrt{3}/\sqrt[4]{2} = b + c\sqrt[4]{2} + d\sqrt{2}$. Compute traces again to show $b = 0$.

   (c) Once more, now show $c = 0$. Then derive a contradiction.

3. (from class) Let K be an algebraic number field and $\mathcal{O}_K$ its ring of integers. Suppose P is a prime ideal in $\mathcal{O}_K$, and $\alpha \in P - P^2$.

   (a) Prove that $\gcd(\alpha\mathcal{O}_K, P^2) = P$.

   (b) Prove that $\operatorname{lcm}(\alpha\mathcal{O}_K, P^2) = \alpha P$

4. Determine all ways a rational prime p could factor in the ring of integers of a degree 3 extension of $\mathbb{Q}$. Include a comment on the inertial degrees and norms of the prime ideals in these factorizations.

5. Let m be a positive integer, K a number field, and $\mathcal{O}_K$ its ring of integers. Show that there are only finitely many ideals I in $\mathcal{O}_K$ such that $N(I) = m$.

6. *Magma* There is a positive integer associated to every number field K called the *class number*. When this number is one, $\mathcal{O}_K$ itself is a unique factorization domain (or the ideals behave just like the elements). Roughly, the class number measures the failure of unique factorization of elements, meaning the larger the number gets, the less the ideals (which always have unique factorization) behave like the elements.

   (a) Use Magma and the command `ClassNumber(K)` to find all imaginary quadratic extensions $K = \mathbb{Q}(\sqrt{-m})$ that have class number 1. (You can stop at $m = 200$–it's not immediately obvious that there even is a bound.) Submit a log file for this part.

   (b) Write a program in Magma to find all imaginary quadratic extensions $K = \mathbb{Q}(\sqrt{-m})$ with class number 2 for $m < 10^5$. Your program should output a list of those m with class number 2. Run the program and submit the program and the output results.