Math 324 Spring 2017
Homework 8
Due: April 19, 2017

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1. Let K be an algebraic number field and $\alpha \in K$. Let β be a conjugate of α relative to K.

   (a) Prove that $D(\alpha) = D(\beta)$.

   (b) Prove that $\text{fld}_K(\alpha) = \text{fld}_K(\beta)$.

   (c) Prove that if $\gamma \in K$ so that $\text{fld}_K(\alpha) = \text{fld}_K(\gamma)$ then α and γ are conjugates relative to K.

2. (a) Prove that $\mathbb{Z} + \mathbb{Z}\sqrt{3} + \mathbb{Z}\sqrt{7} + \mathbb{Z}\sqrt{21}$ is not the ring of integers of $\mathbb{Q}(\sqrt{3}, \sqrt{7})$.

   (b) Compute integral basis and discriminant of $\mathbb{Q}(\sqrt{2}, \sqrt{3})$

3. (From class) Let D be an integral domain and I an ideal of D with K the quotient field of D.

   (a) For any $a \in I$ define $J := \frac{1}{a} \cdot I = \{\frac{1}{a} \cdot x \mid x \in I\} \subseteq K$. Prove that $I \subseteq J$ and if $J \subseteq D$ then J is an ideal of D.

   (b) For all $d \in D$ prove that $\langle d \rangle \cdot I = \{d \cdot x \mid x \in I\}$.

4. Prove that a Dedekind domain D is a UFD if and only if it is a PID.

5. Let I and J be nonzero ideals of a Dedekind domain D. See Purple book Chapter 8 problem #12 for definition of the greatest common divisor and least common multiple of ideals in a Dedekind domain.

   (a) Prove that $\gcd(I, J) = I + J$.

   (b) Prove that $\text{lcm}(I, J) = I \cap J$.

6. Let $I = \langle 2, 1 + \sqrt{-3} \rangle$ in $\mathbb{Z} + \mathbb{Z}\sqrt{-3}$.

   (a) Show that $I \neq \langle 2 \rangle$.

   (b) Show that $I^2 = \langle 2 \rangle \cdot I$.

   (c) Explain what this tells us about unique factorization in $\mathbb{Z} + \mathbb{Z}\sqrt{-3}$.

   (This is more evidence that $\mathbb{Z} + \mathbb{Z}\sqrt{-3}$ should not be the ring of integers of O_K.)