Math 324 Spring 2017
Homework 7
Due: March 17

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1. Prove that the integral closure of $\mathbb{Z}+\mathbb{Z}\sqrt{5}$ in the field $\mathbb{Q}(\sqrt{5}, i) = \{a + bi + c\sqrt{5} + di\sqrt{5} \mid a, b, c, d \in \mathbb{Q}\}$ is
$$\left\{a + ib \mid a, b \in \mathbb{Z} + \mathbb{Z}\left(\frac{1+\sqrt{5}}{2}\right)\right\}.$$

2. (a) Prove that no right triangle with integral sides has area which is a perfect squre.

   (b) Prove that for any right triangle with integral sides, at least one side has length a multiple of 3, at least one side has length a multiple of 4, and at least one side has length a multiple of 5. (One side could satisfy more than one of these conditions.)

3. (a) Find all integer solutions to $x^2 + 2y^2 = z^2$.

   (b) Generalize to $x^2 + py^2 = z^2$ where p is an odd prime.

4. Let $K = \mathbb{Q}(\sqrt{m})$.

   (a) Prove that if $\alpha \in \mathcal{O}_K$ and u any unit on $\mathcal{O}_K$ then $u \mid \alpha$.

   (b) Prove that if $\alpha \in \mathcal{O}_K$ and $\alpha \neq 0$ and α not a unit, then $|\phi_m(\alpha)| > 1$.

   (c) Suppose α is an algebraic number in K. If $m < 0$ show that $\phi_m(\alpha) \geq 0$. Show this is false if $m > 0$.

5. Let $K = \mathbb{Q}(\sqrt{2})$. This problem will prove that the units in $\mathcal{O}_K$ are elements of the form $\pm(1+\sqrt{2})^n$ for any integer n.

   (a) Prove that $\pm(1+\sqrt{2})^n$ are in fact all units.

   (b) Prove that $1+\sqrt{2}$ is the smallest unit > 1 (in the sense of $\mathbb{R}$).

   (c) Fix some other unit v and suppose it is not of the form above. Let n be the integer satisfying $0 \leq (1+\sqrt{2})^{n-1} < v < (1+\sqrt{2})^n$. Use v and some power of $1+\sqrt{2}$ to construct a unit between 1 and $1+\sqrt{2}$.

   (d) Explain the contradiction in (c) and draw the proper conclusion about units in $\mathcal{O}_K$.

6. This problem relies on information we will cover in Monday's class. Part (b) is not required.

   (a) Let m be a cubefree integer. Let $K = \mathbb{Q}(\sqrt[3]{m})$. Determine all the monomorphisms from K to $\mathbb{C}$.

   (b) **Bonus**: Let $\theta = \sqrt[3]{1+i} + \sqrt[3]{1-i}$ and $K = \mathbb{Q}(\theta)$. Determine all the monomorphisms from K to $\mathbb{C}$. This problem requires an understanding of the complex cube root.

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7. *Magma*: The Magma guide for our class has information about how to, given an algebraic number field K, create the ring of integers $\mathcal{O}_K$, as well as ideals in $\mathcal{O}_K$ (see pages 2-3). Use that to help answer the following question.

   I mentioned in class that unique factorization into primes exists for ideals of R, regardless of whether R is a UFD. Use Magma to help you to answer the following question:

   Let m be a squarefree integer and $K = \mathbb{Q}(\sqrt{m})$. Come up with a conjecture about the factorization of the ideal $\langle 2 \rangle$ in $\mathcal{O}_K$ depending on the value of m.

   Work very hard to keep your test cases limited. Plan what you want to test before you start coding and testing. 50 pages of output examples is not more helpful than 1 or 2 carefully constructed pages of output examples.