Math 324 Spring 2017
Homework 5
Due: March 1

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1. Let $n \in \mathbb{Z}$, $n > 1$ and let R be the ring of $n \times n$ matrices with entries from a field F. Let M be the set of $n \times n$ matrices with arbitrary elements of F in the first column, and zeros elsewhere.

   (a) Show that M is a submodule of R when R is considered as a left module over itself.

   (b) Show that M is not a submodule of R when R is considered as a right R-module.

2. Let N be a submodule of an R-module M. Define the *annihilator of N in R* as the set $\{r \in R \mid rn = 0 \text{ for all } n \in N\}$.

   (a) Prove that the annihilator of N in R is an ideal of R.

   (b) Let M be the $\mathbb{Z}$-module $\mathbb{Z}/24\mathbb{Z} \times \mathbb{Z}/15\mathbb{Z} \times \mathbb{Z}/50\mathbb{Z}$. Find the annihilator of M in $\mathbb{Z}$. (Note: $\mathbb{Z}$ is a PID so you should give a generator for the ideal.)

3. Find the minimal polynomial of each of the following algebraic numbers:

$$7, \sqrt[3]{7}, \frac{1+\sqrt[3]{7}}{2}, 1+\sqrt{2}+\sqrt{3}.$$

   Which of these are algebraic integers?

4. (a) Let α be an algebraic number. Show that there is an integer n such that $n\alpha$ is an algebraic integer.

   (b) Show that there exist algebraic numbers of arbitrarily large degree.

5. Let K be a subfield of $\mathbb{C}$ so that $[K : \mathbb{Q}] = 2$.

   (a) Show there is some $u \in \mathbb{Q}$ so that $K = \mathbb{Q}(\sqrt{u})$.

   (b) Show that for any a and b in $\mathbb{Z}$ with $b \neq 0$ that $\mathbb{Q}(\sqrt{\frac{a}{b}}) = \mathbb{Q}(\sqrt{ab})$.

   (c) Prove that there is an integer d such that $K = \mathbb{Q}(\sqrt{d})$ where d is not divisible by the square of any prime.

6. *Magma* The Magma guide includes information about how to create simple field extensions. In class we will prove exactly what the elements of $\mathbb{Q}(\sqrt{m})$ which are algebraic integers look like.

   Test elements of $\mathbb{Z} + \mathbb{Z}\sqrt{m}$ and $\mathbb{Z} + \mathbb{Z}\left(\frac{1+\sqrt{m}}{2}\right)$ for various m both positive and negative to determine if they are algebraic integers. The Magma command `IsIntegral(a)` will determine if a is an algebraic integer.

   Then come up with a conjecture for any m for which elements of $\mathbb{Q}(\sqrt{m})$ are algebraic integers.