Math 324 Spring 2017
Homework 4
Due: February 22, 2017

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1. Find an example of UFD D where some $a \in D$ is irreducible but $\langle a \rangle$ is not maximal.

2. Prove that the ring of all continuous real valued functions on $[0, 1]$ is not Noetherian by exhibiting an explicit infinite ascending chain of ideals.

3. (a) Prove that if R is a Noetherian ring, and I is an ideal in R, then R/I is a Noetherian ring.

   (b) Prove the converse of Hilbert's basis theorem: If the polynomial ring $R[x]$ is a Noetherian ring then R is a Noetherian ring.

4. Call a vector space V *Noetherian* if every ascending chain of subspaces of V terminates. Prove that $\mathbb{R}^n$ is Noetherian for all positive integers n.

5. Let M be an R-module and N a submodule of M.

   (1) If M_1 is another submodule of M, show that $M_1 \cap N$ is a submodule of N.

   (2) If M_1 is another submodule of M, show that $\overline{M_1} := \{m + N \in M/N \mid m \in M_1\}$ is a submodule of M/N.

   (3) If $\overline{M_1}$ is a submodule of M/N, show that $\{m \in M \mid m + N \in \overline{M_1}\}$ is a submodule of M.

6. Define a set $\mathrm{Hom}(M, N)$ to be the set of R-module homomorphisms from M to N.

   (a) Prove that this set is an abelian group under addition of homomorphism.

   (b) If R is also commutative, prove that the set of homomorphisms is an R module under the R-action $(r \cdot \phi)(m) = r \cdot (\phi(m))$ for all $m \in M$ where $\phi(m)$ is a homomorphism from M to N.

7. *Magma*: Let m be an odd number, $(a, m) = 1$. Come up with a conjecture for the number of solutions to the congruence $x^2 \equiv a \bmod m$. We only count solutions x in the set $\{0, 1, \ldots, m-1\}$ since, if x is a solution, anything congruent to x mod m is also a solution. You might want to consider the cases where m is a prime or a power of a prime first, then consider the factorization of m into primes.

8. *Challenge*: (We discussed part of (c) in class.) Let $R = \mathbb{Z} + x\mathbb{Q}[x]$ be the set of polynomials in x with rational coefficients whose constant term is an integer.

   (a) Prove that R is an integral domain and its units are ± 1.

   (b) Show that the irreducibles in R are $\pm p$ where p is a prime in $\mathbb{Z}$ and the polynomials $f(x)$ that are irreducible in $\mathbb{Q}[x]$ and have a constant term ± 1. Prove that these irreducibles are prime in R.

   (c) Show that x cannot be written as the product of irreducibles in R (in particular x is not irreducible) and conclude that R is not a UFD.

   (d) Show that x is not a prime in R and describe the quotient ring $R/\langle x \rangle$.