Math 324 Spring 2017
Homework 2
Due: February 8, 2017

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1. Let m be an integer with $m \equiv 1 \bmod 4$ and $m < -3$. Prove that
$$U\left(\mathbb{Z} + \mathbb{Z}\left(\frac{1+\sqrt{m}}{2}\right)\right) = \{\pm 1\}.$$
(Hint: Consider complex conjugates.)

2. (a) Prove that $\sqrt{10}$ is not a prime in $\mathbb{Z} + \mathbb{Z}\sqrt{10}$.
   (b) Prove that $\sqrt{10}$ is irreducible in $\mathbb{Z} + \mathbb{Z}\sqrt{10}$.
   (c) Prove that $\mathbb{Z} + \mathbb{Z}\sqrt{10}$ is not a PID.
   (d) Give an example of an ideal in $\mathbb{Z} + \mathbb{Z}\sqrt{10}$ that is not principal.

3. Prove that the quotient of a Principal Ideal Domain by a prime ideal is again a Principal Ideal Domain.

4. (a) Is 2 a prime in $\mathbb{Z} + \mathbb{Z}i$? What about 5?
   (b) Prove that $\langle 2, 1+\sqrt{-5}\rangle$ and $\langle 3, 1-\sqrt{-5}\rangle$ are prime ideals in $\mathbb{Z} + \mathbb{Z}\sqrt{-5}$.

5. Let D be the integral domain $\mathbb{Z} + \mathbb{Z}i$.
   (a) Prove that for every $n \in \mathbb{Z}^+$ there are only a finite number of elements $\alpha \in D$ such that $\phi_{-1}(\alpha) \leq n$.
   (b) Given a nonzero ideal $I = \langle a \rangle \in D$ prove that every coset of I is represented by an element of norm less than $\phi_{-1}(a)$.
   (c) Prove that the quotient ring $(\mathbb{Z} + \mathbb{Z}i)/I$ is finite for any nonzero ideal I of $\mathbb{Z} + \mathbb{Z}i$.

6. For any commutative ring R with identity we define a *least common multiple* of two nonzero elements a and b in R to be an element c in R so that (i) $a \mid c$ and $b \mid c$ and (ii) if $a \mid d$ and $b \mid d$ then $c \mid d$.

   For this problem, assume D is a Principal Ideal Domain.
   (a) Prove that a least common multiple of a and $b \in D$ is a generator for the ideal $\langle a \rangle \cap \langle b \rangle$. (Hence any two nonzero elements of D have a least common multiple.)
   (b) Prove that in a Euclidean Domain a least common multiple of a and b is $\frac{ab}{(a,b)}$.

7. *Challenge*: This problem is extra credit and not required. You will need to do some reading about a result called *Zorn's Lemma*. The goal is to prove that if D is an integral domain where every prime ideal is principal, then D is a PID.

   (a) Assume the set of ideals of D that are not principal is nonempty and prove that this set has a maximal element under inclusion.

(b) Let I be an ideal which is maximal with respect to being nonprincipal and let a, $b \in D$ with $ab \in I$ but $a \notin I$ and $b \notin I$. Let $I_a = \langle I, a \rangle$ be the ideal generated by I and a and let $I_b = \langle I, b \rangle$ be the ideal generated by I and b, and define $J = \{r \in D \mid rI_a \subseteq I\}$. Prove that $I_a = \langle c \rangle$ and $J = \langle d \rangle$ are principal ideals in D with $I \subsetneqq I_b \subseteq J$ and $I_a J = \langle cd \rangle \subseteq I$.

(c) If $x \in I$ show that $x = sc$ for some $s \in J$. Deduce that $I = I_a J$ is principal.

(d) Conclude that D is a PID.