Math 324 Spring 2017
Homework Last!
Due: May 12, 2017

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1. Assume that $O_{K_m} = \mathbb{Z}[\zeta_m]$, although we didn't prove this for arbitrary positive integers m. Prove that $1 + \zeta_m + \zeta_m^2 + \cdots + \zeta_m^{k-1}$ is a unit in O_{K_m} if k is a positive integer relatively prime to m.

2. (a) In $K_3 = \mathbb{Q}(\zeta_3)$ prove that the norm of $\alpha \in O_K$ is of the form $\frac{1}{4}(a^2 + 3b^2)$ where a and b are rational integers which are either both even or both odd.

   (b) Use part (a) to deduce that there are precisely six units in O_K and find them all. (You may NOT use (c) for this part.)

   (c) K_3 is the same field as $\mathbb{Q}(\sqrt{-3})$ (convince yourself of this). Relate your answer in (b) to what we know about units in $\mathbb{Q}(\sqrt{-3})$.

3. (6 points) In this problem, you will find the class number for $K = \mathbb{Q}(\sqrt{-5})$ using the Theorem from class on May 3.

   (a) Compute the λ for this case, as in the proof on May 3.

   (b) Determine the decomposition of all ideals $\langle p \rangle$ so that p is a rational prime less than the λ you found in (a).

   (c) If you did everything correctly in (a) and (b), several of these ideals should look familiar from old homeworks or class examples. Show that the ideal(s) over $\langle 5 \rangle$ are principal. For the rest of the problem, you may assume the other ideals you found in (a) are non-principal ideal.

   (d) Determine the order of the ideal class containing one of the prime ideals over $\langle 2 \rangle$.

   (e) Show that the prime ideal(s) over $\langle 3 \rangle$ are in the same ideal class as those over $\langle 2 \rangle$. The same result is true for the prime ideal(s) over $\langle 7 \rangle$, but you can assume this for the rest of the problem.

   (f) Determine the class number for K.

4. Let D be a Dedekind Domain, and let I be an nonzero ideal of D. Fix a nonzero $\alpha \in I$.

   (a) Use the prime decompositions of I and $\langle \alpha \rangle$ to construct a $\beta \in D$ so that $\gcd(\langle \alpha \rangle, \langle \beta \rangle) = I$. (Suggestion: The Chinese Remainder Theorem might come in handy.)

   (b) Use part (a) to prove that in a Dedekind Domain, any nonzero ideal can be generated by 2 elements.