

Math 324 Spring 2017
Homework 10
Due: May 3, 2017

- Factor $\langle 2 \rangle$ into prime ideals in O_K where $K = \mathbb{Q}(\sqrt{47})$.
 - Factor $\langle 6 \rangle$ into prime ideals in O_K where $K = \mathbb{Q}(\sqrt{366})$.
- Prove that $x^2 - my^2 = -1$ has no solution if $m \equiv 3 \pmod{4}$.
 - It's not hard to see that $x^2 - 80y^2 = 1$ has a solution (what is it?). Use the next problem to prove that $x^2 - 80y^2 = -1$ has no solution in integers.
- Assuming that $x^2 - my^2 = -1$ is solvable, let x_1, y_1 be the smallest solution greater than 1. Prove that x_2 and y_2 defined by $x_2 + y_2\sqrt{m} = (x_1 + y_1\sqrt{m})^2$ is the smallest solution greater than 1 of $x^2 - my^2 = 1$. (This is Number Theory, solution means integer solution!)
- If we let $\alpha = 3 + 4\sqrt{-1}$ then the equation $N(\alpha) = 5^2$ is another way of expressing the fact that $(3, 4, 5)$ is a Pythagorean triple.
 - Use the multiplicativity of norms, $N(\alpha^n) = (N(\alpha))^n$ for $n = 2$ and $n = 3$ to find two other triples.
 - Let $\beta = 5 + 12\sqrt{-1}$ so that $N(\beta) = 13^2$. Use α and β to find a triple with hypotenuse 65.
 - Using $\gamma = 12 + 5\sqrt{-1}$ instead, find another triple with hypotenuse 65.
- Prove that the following assertion is false in $\mathbb{Q}(i)$: If $N(\alpha)$ is a rational integer, then α is an algebraic integer.
 - Prove that the above assertion is false in every quadratic field. (A suggestion: Consider $\frac{x+2\sqrt{m}}{y}$ for a particular x and y value and see (a) for one of several special cases.)
- Magma* There's a famous story about an English mathematician named Hardy and an Indian mathematician named Ramanujan. Hardy was visiting Ramanujan in the hospital and complained that the number of the taxicab he had driven to the hospital in was a boring number. Ramanujan countered that the number was actually very interesting as it was the smallest positive number that could be written as the sum of two integer cubes in two different ways. Use *Magma* to find Hardy's taxi number. You must include your full log file for credit.