Math 324 Spring 2017
Homework 1
Due: February 1, 2017

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1. (a) Given a prime p, prove that if a has order 3 in $\mathbb{Z}/p\mathbb{Z}$ then $a^2 + a + 1 \equiv 0 \bmod p$.
   (b) Generalize (a) to arbitrary order. Also, can we remove the assumption that p is prime?

2. Prove that every finite integral domain is a field.

3. Prove that, using notation from the Division Algorithm, $(a, b) = (b, r)$.

4. Let A and B be ideals of an integral domain D.
   (a) Prove that $A \cap B$ is also an ideal of D.
   (b) Prove that $AB \subseteq A \cap B$.
   (c) Prove that $(A \cap B)(A + B) \subseteq AB$.
   (d) Give an example to show that equality does not always hold in (c).

5. We say two ideals A and B in a ring R are *comaximal* if $A + B = R$. Prove that in a Principal Ideal Domain D, two ideals $\langle a \rangle$ and $\langle b \rangle$ are comaximal if and only if a greatest common divisor of a and b is the 1 in D.

6. (a) Let D be an integral domain. Let a, b, and $c \in D$ be such that $\langle a, c \rangle = D$. Prove that $\langle a, bc \rangle = \langle a, b \rangle$.
   (b) In the special case where $D = \mathbb{Z}$, restate (a) in terms of gcd's and prove your restatement using elementary methods.

7. Find an integral domain without any irreducible elements.

8. *Magma*: Define $\left(\frac{a}{p}\right)$ to be 1 if $x^2 \equiv a \bmod p$ has a non-zero solution x, -1 if $x^2 \equiv a \bmod p$ does not have a solution x, and 0 if $a \mid p$. This symbol is called the Legendre Symbol and we will use it next week. The Magma command for this function is `LegendreSymbol(a,p)`. Use Magma to formulate conjectures for the following problems.
   (a) For any odd prime p, conjecture what the value of $\left(\frac{2}{p}\right)$ is.
   (b) For any odd primes p and q, come up with a conjecture relating the values of $\left(\frac{p}{q}\right)$ and $\left(\frac{q}{p}\right)$.